

The Proof of Innocence

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We show that if a car stops at a stop sign, an observer, e.g., a police officer, located at a certain distance perpendicular to the car trajectory, must have an illusion that the car does not stop, if the following three conditions are satisfied: (1) the observer measures not the linear but angular speed of the car; (2) the car decelerates and subsequently accelerates relatively fast; and (3) there is a short-time obstruction of the observer's view of the car by an external object, e.g., another car, at the moment when both cars are near the stop sign.

I. INTRODUCTION

It is widely known that an observer measuring the speed of an object passing by, measures not its actual linear velocity by the angular one. For example, if we stay not far away from a railroad, watching a train approaching us from far away at a constant speed, we first perceive the train not moving at all, when it is really far, but when the train comes closer, it appears to us moving faster and faster, and when it actually passes us, its visual speed is maximized.

This observation is the first building block of our proof of innocence. To make this proof rigorous, we first consider the relationship between the linear and angular speeds of an object in the toy example where the object moves at a constant linear speed. We then proceed to analyzing a picture reflecting what really happened in the considered case, that is, the case where the linear speed of an object is not constant, but what is constant instead is the deceleration and subsequent acceleration of the object coming to a complete stop at a point located closest to the observer on the object's linear trajectory. Finally, in the last section, we consider what happens if at that critical moment the observer's view is briefly obstructed by another external object.

II. CONSTANT LINEAR SPEED

Consider Fig. 1 schematically showing the geometry of the considered case, and assume for a moment that C 's linear velocity is constant in time t ,

$$v(t) \equiv v_0. \quad (1)$$

Without loss of generality we can choose time units t such that $t = 0$ corresponds to the moment when C is at S . Then distance x is simply

$$x(t) = v_0 t. \quad (2)$$

Observer O visually measures not the linear speed of C but its angular speed given by the first derivative of angle α with respect to time t ,

$$\omega(t) = \frac{d\alpha}{dt}. \quad (3)$$

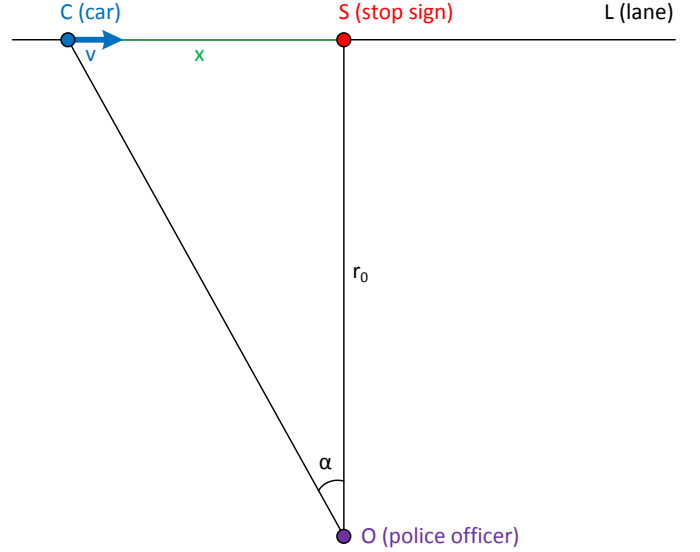


FIG. 1: The diagram showing schematically the geometry of the considered case. Car C moves along line L . Its current linear speed is v , and the current distance from stop sign S is x , $|CS| = x$. Another road connects to L perpendicularly at S . Police officer O is located on that road at distance r_0 from the intersection, $|OS| = r_0$. The angle between OC and OS is α .

To express $\alpha(t)$ in terms of r_0 and $x(t)$ we observe from triangle OCS that

$$\tan \alpha(t) = \frac{x(t)}{r_0}, \quad (4)$$

leading to

$$\alpha(t) = \arctan \frac{x(t)}{r_0}. \quad (5)$$

Substituting the last expression into Eq. (3) and using the standard differentiation rules there, i.e., specifically the fact that

$$\frac{d}{dt} \arctan f(t) = \frac{1}{1+f^2} \frac{df}{dt}, \quad (6)$$

where $f(t)$ is any function of t , but it is $f(t) = v_0 t / r_0$ here, we find that the angular speed of C that O observes

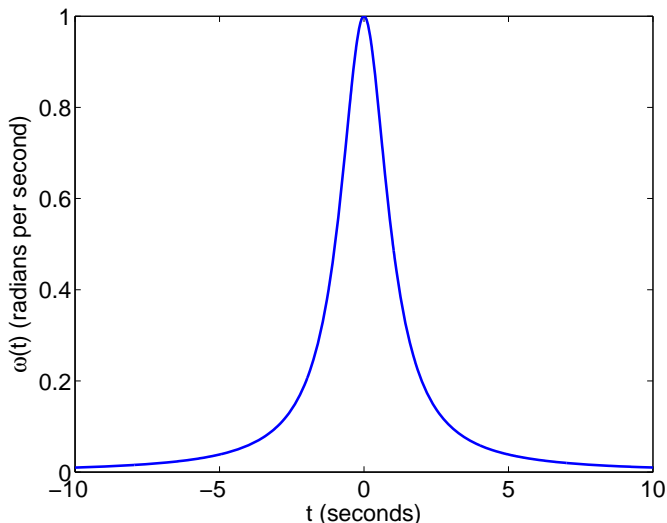


FIG. 2: The angular velocity ω of C observed by O as a function of time t if C moves at constant linear speed v_0 . The data is shown for $v_0 = 10$ m/s = 22.36 mph and $r_0 = 10$ m = 32.81 ft.

as a function of time t is

$$\omega(t) = \frac{v_0/r_0}{1 + \left(\frac{v_0}{r_0}\right)^2 t^2}. \quad (7)$$

This function is shown in Fig. 2. It confirms and quantifies the observation discussed in the previous section, that at O , the visual angular speed of C moving at a constant linear speed is not constant. It is the higher, the closer C to O , and it goes over a sharp maximum at $t = 0$ when C is at the closest point S to O on its linear trajectory L .

III. CONSTANT LINEAR DECELERATION AND ACCELERATION

In this section we consider the situation closely mimicking what actually happened in the considered case. Specifically, C , instead of moving at constant linear speed v_0 , first decelerates at constant deceleration a_0 , then comes to a complete stop at S , and finally accelerates with the same constant acceleration a_0 .

In this case, distance $x(t)$ is no longer given by Eq. (2). It is instead

$$x(t) = \frac{1}{2} a_0 t^2. \quad (8)$$

If this expression does not look familiar, it can be easily derived. Indeed, with constant deceleration/acceleration, the velocity is

$$v(t) = a_0 t, \quad (9)$$

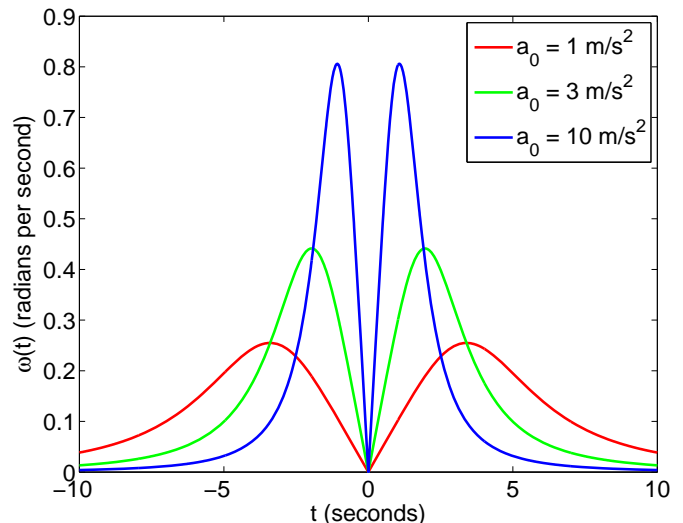


FIG. 3: The angular velocity ω of C observed by O as a function of time t if C moves with constant linear deceleration a_0 , comes to a complete stop at S at time $t = 0$, and then moves with the same constant linear acceleration a_0 . The data are shown for $r_0 = 10$ m.

but by the definition of velocity,

$$v(t) = \frac{dx}{dt}, \quad (10)$$

so that

$$dx = v(t) dt. \quad (11)$$

Integrating this equation we obtain

$$x(t) = \int_0^x dx = \int_0^t v(t) dt = a_0 \int_0^t t dt = \frac{1}{2} a_0 t^2. \quad (12)$$

Substituting the last expression into Eq. (5) and then differentiating according to Eq. (3) using the rule in Eq. (6) with $f(t) = a_0 t^2 / (2r_0)$, we obtain the angular velocity of C that O observes

$$\omega(t) = \frac{\frac{a_0}{r_0} t}{1 + \frac{1}{4} \left(\frac{a_0}{r_0}\right)^2 t^4}. \quad (13)$$

This function is shown in Fig. 3 for different values of a_0 . In contrast to Fig. 2, we observe that the angular velocity of C drops to zero at $t = 0$, which is expected because C comes to a complete stop at S at this time. However, we also observe that the higher the deceleration/acceleration a_0 , the more similar the curves in Fig. 3 become to the curve in Fig. 2. In fact, the blue curve in Fig. 3 is quite similar to the one in Fig. 2, *except the narrow region* between the two peaks in Fig. 3, where the angular velocity quickly drops down to zero, and then quickly rises up again to the second maximum.

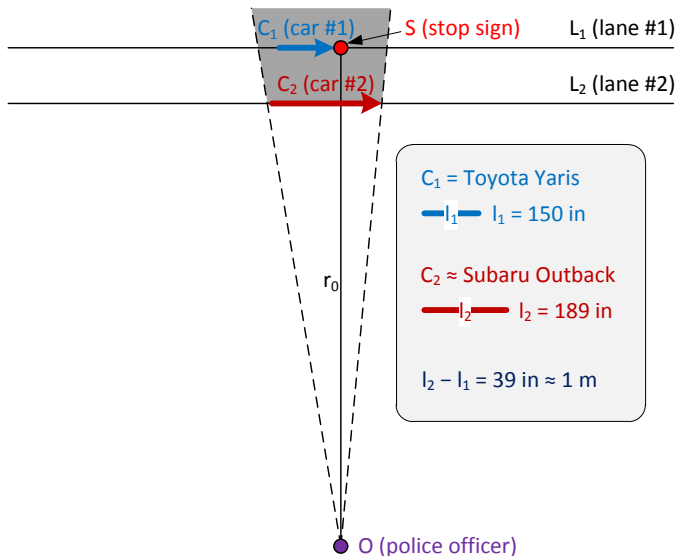


FIG. 4: The diagram showing schematically the brief obstruction of view that happened in the considered case. The O 's observations of car C_1 moving in lane L_1 are briefly obstructed by another car C_2 moving in lane L_2 when both cars are near stop sign S . The region shaded by the grey color is the area of poor visibility for O .

IV. BRIEF OBSTRUCTION OF VIEW AROUND $t = 0$.

Finally, we consider what happens if the O 's observations are briefly obstructed by an external object, i.e., another car, see Fig. 4 for the diagram depicting the considered situation. The author/defendant (D.K.) was driving Toyota Yaris (car C_1 in the diagram), which is one of the shortest cars available on the market. Its length is $l_1 = 150$ in. (Perhaps only the Smart Cars are shorter?) The exact model of the other car (C_2) is unknown, but it was similar in length to Subaru Outback, whose exact length is $l_2 = 189$ in.

To estimate times t_p and t_f at which the partial and, respectively, full obstructions of view of C_1 by C_2 began and ended, we must use Eq. (8) substituting there $x_p = l_2 + l_1 = 8.16$ m, and $x_f = l_2 - l_1 = 0.99$ m, respectively. To use Eq. (8) we have to know C_1 's deceleration/acceleration a_0 . Unfortunately, it is difficult to measure deceleration or acceleration without special tools, but we can roughly estimate it as follows. D.K. was badly sick with cold on that day. In fact, he was sneezing while approaching the stop sign. As a result he involuntarily pushed the brakes very hard. Therefore we can assume that the deceleration was close to maximum possible for a car, which is of the order of $10 \text{ m/s}^2 = 22.36 \text{ mph/s}$. We will thus use $a_0 = 10 \text{ m/s}^2$. Substituting these values of a_0 , x_p , and x_f into Eq. (8) inverted for t ,

$$t = \sqrt{\frac{2x}{a_0}}, \quad (14)$$

we obtain

$$t_p = 1.31 \text{ s}, \quad (15)$$

$$t_f = 0.45 \text{ s}. \quad (16)$$

The full durations of the partial and full obstructions are then just double these times.

Next, we are interested in time t' at which the angular speed of C_1 observed by O without any obstructions goes over its maxima, as in Fig. 3. The easiest way to find t' is to recall that the value of the first derivative of the angular speed at t' is zero,

$$\frac{d\omega}{dt} = \dot{\omega}(t') = 0. \quad (17)$$

To find $\dot{\omega}(t)$ we just differentiate Eq. (13) using the standard differentiation rules, which yield

$$\dot{\omega}(t) = 4 \frac{a_0}{r_0} \frac{1 - \frac{3}{4} \left(\frac{a_0}{r_0}\right)^2 t^4}{\left[1 + \frac{1}{4} \left(\frac{a_0}{r_0}\right)^2 t^4\right]^2}. \quad (18)$$

This function is zero only when the numerator is zero, so that the root of Eq. (17) is

$$t' = \sqrt[4]{\frac{4}{3}} \sqrt{\frac{r_0}{a_0}}. \quad (19)$$

Substituting the values of $a_0 = 10 \text{ m/s}^2$ and $r_0 = 10$ m in this expression, we obtain

$$t' = 1.07 \text{ s}. \quad (20)$$

We thus conclude that time t' lies between t_f and t_p ,

$$t_f < t' < t_p, \quad (21)$$

and that differences between all these times is actually quite small, compare Eqs. (15,16,20).

These findings mean that the angular speed of C_1 as observed by O went over its maxima when the O 's view of C_1 was partially obstructed by C_2 , and very close in time to the full obstruction. In lack of complete information, O interpolated the available data, i.e., the data for times $t > t' \approx t_f \approx t_p$, using the simplest and physiologically explainable linear interpolation, i.e., by connecting the boundaries of available data by a linear function. The result of this interpolation is shown by the dashed curve in Fig. 5. It is remarkably similar to the curve showing the angular speed of a hypothetical object moving at constant speed $v_0 = 8 \text{ m/s} \approx 18 \text{ mph}$.

V. CONCLUSION

In summary, police officer O made a mistake, confusing the real spacetime trajectory of car C_1 —which moved at approximately constant linear deceleration, came to a

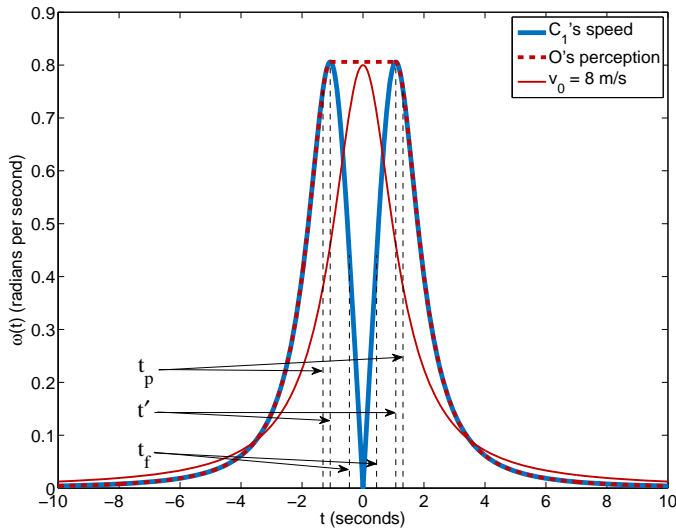


FIG. 5: The real angular speed of C_1 is shown by the blue solid curve. The O 's interpolation is the dashed red curve. This curve is remarkably similar to the red solid curve, showing the angular speed of a hypothetical object moving at constant linear speed $v_0 = 8 \text{ m/s} = 17.90 \text{ mph}$.

complete stop at the stop sign, and then started moving again with the same acceleration, the blue solid line in Fig. 5—for a trajectory of a hypothetical object moving at approximately constant linear speed without stopping at the stop sign, the red solid line in the same figure. However, this mistake is fully justified, and it was made possible by a combination of the following three factors:

1. O was not measuring the linear speed of C_1 by any special devices; instead, he was estimating the visual angular speed of C_1 ;
2. the linear deceleration and acceleration of C_1 were relatively high; and
3. the O 's view of C_1 was briefly obstructed by another car C_2 around time $t = 0$.

As a result of this unfortunate coincidence, the O 's perception of reality did not properly reflect reality.